Epistemic logic and the Logical Omniscience Problem

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Outline

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- Von Wright's logical system $M(T)$.

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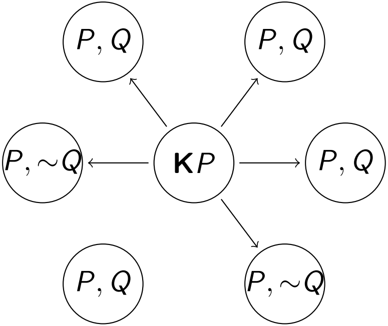
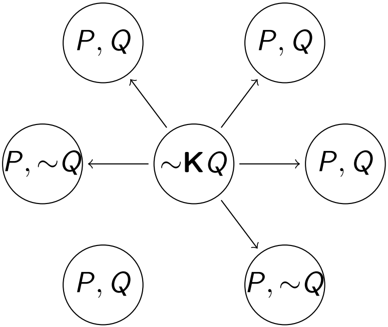
- $\mathbf{K}(\phi \to \psi) \to (\mathbf{K}\phi \to \mathbf{K}\psi)$
- $\mathbf{K}\phi \to \phi$

- $\mathbf{K}\phi \rightarrow \mathbf{K}(\mathbf{K}\phi)$
- $\sim\mathbf{K}\phi \rightarrow \mathbf{K}(\sim\mathbf{K}\phi)$

- Hittikka's semantic analysis of epistemic concepts
- the old adage that "knowledge is the elimination of the uncertainty"



epistemic alternatives (possible worlds)
alternativeness (accessibility) relation



epistemic alternatives (possible worlds)
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S4 (System for Hittikka's models for knowledge)

Classical propositional axiom schemes

$\mathbf{K}(\phi \to \psi) \to (\mathbf{K}\phi \to \mathbf{K}\psi)$

$\mathbf{K}\phi \to \phi$

$\mathbf{K}\phi \to \mathbf{K}(\mathbf{K}\phi)$

$\vdash \psi$, if $\vdash \phi \to \psi$ and $\vdash \phi$

$\vdash \mathbf{K}\phi$, if $\vdash \phi$

KD4 (System for Hittikka's models for belief)

Classical propositional axiom schemes

$\mathbf{K}(\phi \rightarrow \psi) \rightarrow (\mathbf{K}\phi \rightarrow \mathbf{K}\psi)$

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- Von Wright's syntactical proof system
- Hinttika's possible world semantics
- KK-thesis
- Internalism vs. Externalism
- Quantified Epistemic Logic
- Moore's Paradox
- Knowability Paradox
- Entailment thesis
- Deductive Closure Principle
- Logical Omniscience Problem

Two phases

- Philosophical investigation into the formal formulation of epistemic concepts
- Applications in Computer Science
  - A.I.
  - Distributed Systems

S5

Classical propositional axiom schemes

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Outline

The problem

- $\dfrac{\vdash \phi \to \psi}{\vdash \mathbf{K}\phi \to \mathbf{K}\psi}$ (Rule of Logical Consequence Closure)
- Undesirable features
  - Knowing all axioms of set theory implies knowing all mathematics
  - Players of a chess game know the winning strategy of the game from the beginning.
- Approaches are proposed for dealing with the problem.

Logical omniscience is not a problem

- External Knowledge:
- Example of applications: the analysis of distributed systems.

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- $$\frac{\vdash \phi \rightarrow \psi}{\vdash \mathbf{K}\phi \rightarrow \mathbf{K}\psi}$$
- Implicit Knowledge: what the agent *potentially* knows at the moment of the investigation.

Logical omniscience is not a problem

- $$\frac{\vdash \phi \rightarrow \psi}{\vdash \mathbf{K}\phi \rightarrow \mathbf{K}\psi}$$
- Implicit Knowledge: what the agent *potentially* knows at the moment of the investigation.
- Limitation of knowledge: $\sim\mathbf{K}(\phi \& \sim\mathbf{K}\phi)$

Logical omniscience is not a problem

- Explicit Knowledge:
- This is the concept that the first epistemic logic tries to capture, and the one we want to use to predict the behavior of an epistemic agent (a human being, or a robot), since this the epistemic concept will affect agent's behavior.

Logical omniscience is not a problem

- Explicit Knowledge:
- This is the concept that the first epistemic logic tries to capture, and the one we want to use to predict the behavior of an epistemic agent (a human being, or a robot), since this the epistemic concept will affect agent's behavior.
- And this is the type of epistemic logic suffering from the problem of logical omniscience.

Not a problem of logical omniscience

- Deductive Closure Principle: $\mathbf{K}(\phi \to \psi) \to (\mathbf{K}\phi \to \mathbf{K}\psi)$ does not lead to the problem of logical omniscience.
- Rule of Logical Consequence Closure $\dfrac{\vdash \phi \to \psi}{\vdash \mathbf{K}\phi \to \mathbf{K}\psi}$

Outline

- Belief set approach
- $\mathbf{K}(A\&B)$ but $\sim\mathbf{K}(B\&A)$?

- Montague-Scott's neighborhood semantics approach.
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- The modeled agent is still logical omniscient.

- Belief set approach
- Montague-Scott's neighborhood semantics approach.
- "Epistemic logic is either not epistemic, or not logic at all"

Other Approaches

Other approaches:

- Impossible world semantics approach.
- Deductive model approach.
- Awareness function approach.

Other Approaches

- Deductive model approach.
- $$\frac{D(k) \wedge \phi \qquad D(l) \wedge (\phi \to \psi)}{D(k{+}l{+}1) \wedge \psi}, \quad \text{for } k{+}l{+}1 \leq n,$$

  where, e.g., $D(k)$ means ϕ is derived through k times of applications of logical rules from the base formulas.

Other Approaches

- First, the deductive rationality of the agent is not fully represented. (Consider the moment when $k+l+1 = n+1$ in the Deductive Model)
- Second, the relation between the lack of the resources of reasoning and the model is not clear.
- Third, when the bound of the resource is adjusted, new model is needed.

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tS4($\mathcal{A}$) Axiom Systems

classical propositional axiom schemes	
$\mathbf{K}^i(\phi \to \psi) \to (\mathbf{K}^j\phi \to \mathbf{K}^k\psi) \quad i,j < k$	(Deduction by Modus Ponens)
$\mathbf{K}^iA \to \mathbf{K}^j(\mathbf{K}^iA) \quad i < j$ if $A \in \mathcal{A}$	(Deduction by $\mathcal{A}$-Epistemization)
$\mathbf{K}^i\phi \to \mathbf{K}^j\phi \quad i < j$	(Monotonicity)
$\mathbf{K}^i\phi \to \mathbf{K}^j(\mathbf{K}^i\phi) \quad i < j$	(Positive Introspection)
$\mathbf{K}^i\phi \to \phi$	(Truth Axiom)
$\vdash \psi$, if $\vdash \phi \to \psi$ and $\vdash \phi$	(Modus Ponens)
$\vdash \mathbf{K}^iA$, if $A \in \mathcal{A}$ and $f(A) \leq i$	($\mathcal{A}$-Epistemization)

- Temporalization Theorem

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- Given a *full* tS4-logical base $\mathcal{A}$, ϕ is S4 valid (a S4 theorem) if and only if we can find suitable numerical labels *i* for each modal subformulas of the form $\mathbf{K}\psi$ such that when we substitute $\mathbf{K}^i\psi$ for $\mathbf{K}\psi$, the result formula ϕ^τ is S4($\mathcal{A}$) valid (a S4($\mathcal{A}$) theorem)

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- $\mathbf{K}(\phi \to \psi) \to (\mathbf{K}\phi \to \mathbf{K}\psi)$
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- $\mathbf{K}(\phi \to \psi) \to (\mathbf{K}\phi \to \mathbf{K}\psi)$
- $\mathbf{K}^{21}(\phi \to \psi) \to (\mathbf{K}^{12}\phi \to \mathbf{K}^{22}\psi)$
- The Theorem shows that there is indeed a temporal structure hidden in the setting of MEL, but only revealed in the framework of tMEL.

	R	$\mathfrak{A}$
tK	no condition	normal
tKT	reflexive	normal
$tK4$	transitive	positive regular
$tK5$	euclidean	negative regular
$tKT5$	reflexive, and euclidean	negative regular
$tK45$	transitive and euclidean	positive and negative regular
$tKT45$	transitive, reflexive, euclidean	positive and negative regular

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Language of MEL

Constructors:

- propositional letters, P, Q, R, ...
- boolean connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$
- epistemic operator **K**

Formulas ($\mathcal{L}_{MEL}$):

- propositional letters
- $\phi \rightarrow \psi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, and
- $\mathbf{K}\phi$.

- **K**ϕ means
  - the agent knows ϕ, or
  - ϕ is known (by the agent)

Language of tMEL

Constructors:

- propositional letters, P, Q, R, $\ldots$
- boolean connectives $\rightarrow$, $\wedge$, $\vee$, $\neg$
- epistemic operator **K**

Formulas ($\mathcal{L}_{tMEL}$):

- propositional letters
- $\phi \rightarrow \psi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\neg\phi$, and
- $\mathbf{K}^i\phi$, for $i \in \mathbb{N}$

- $\mathbf{K}^i\phi$ means
  - the agent knows ϕ at time i, or
  - ϕ is known (by the agent) at i

- Basic idea:
  - Every tMEL agent is regarded as employing some kind of axiomatic reasoning.
  - The reasoning process will be recorded by a machinery called *awareness function*.

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- In a way, $\alpha(\phi)=i$ means that the agent needs to take i steps (i units of time) to produce the shortest proof of ϕ.

- Each awareness function will be based on a tuple $\mathcal{A} = \langle \mathbf{A}, f \rangle$, where $\mathbf{A}$ is a set of formulas, and $f : \mathbf{A} \to \mathbb{N}$.

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- The formulas in $\mathbf{A}$ are functioning as the axioms of the agent's deductive reasoning.
- The agent is supposed to be aware of these formulas inherently, or hears them from someone else.
- The function f denotes when the agent is aware of these formulas by the above methods.

tS4($\mathcal{A}$)-Awareness function

A partial function $\alpha : \mathcal{L}_{tMEL} \to \mathbb{N}$ is an $\mathcal{A}$-awareness function if it satisfies

- if $A \in \mathbf{A}$, then $\alpha(A) \leq f(A)$. (*Initial Condition*)
- if $\alpha(\phi \to \psi)\downarrow$ and $\alpha(\phi)\downarrow$, then

  $\alpha(\psi) \leq \max(\alpha(\phi \to \psi), \alpha(\phi)) + 1$ (*Deduction by Modus Ponens*)
- if $A \in \mathbf{A}$ and $f(A) = i$, then

  $\alpha(\mathbf{K}^i A) \leq i + 1$, (*Deduction by $\mathcal{A}$-Epistemization*)
- for any ϕ, if $\alpha(\phi) = i$, then

  $\alpha(\mathbf{K}^i \phi) \leq i + 1$, (*Inner Positive Introspection*)

S4 Structure

$M = \langle W, \mathcal{R}, \mathcal{V} \rangle$

- W a nonempty set of worlds
- $\mathcal{R}$ a reflexive and transitive relation on W
- $\mathcal{V}$ a truth evaluation of propositional letters in each world $w \in W$.

tS4($\mathcal{A}$)-Structure

$M = \langle W, R, \mathfrak{A}, \mathcal{V} \rangle$

- $\langle W, R, \mathcal{V} \rangle$ is an S4 structure,
- $\mathfrak{A}$ is a collection of tS4($\mathcal{A}$)-awareness functions α_w indexed by the worlds $w \in W$
- $\mathfrak{A}$ is monotonic:
  $\alpha_{w'}(\phi) \leq \alpha_w(\phi)$ for $\phi \in \mathcal{L}_{tMEL}$, and $w\mathcal{R}w'$.

Modal Epistemic Logic

the truth of MEL formulas is defined as:

- $(M, w) \Vdash P \quad P$ is a propositional letter.
- $(M, w) \Vdash F \vee G$ iff $(M, w) \Vdash F$, or $(M, w) \Vdash G$
- $(M, w) \Vdash F \wedge G$ iff $(M, w) \Vdash F$, and $(M, w) \Vdash G$
- $(M, w) \Vdash F \rightarrow G$ iff $(M, w) \nVdash F$, or $(M, w) \Vdash G$
- $(M, w) \Vdash \neg F$ iff $(M, w) \nVdash F$
- $(M, w) \Vdash \mathbf{K}F$ iff for all w', $w\mathcal{R}w'$ $(M, w') \Vdash F$

Timed Modal Epistemic Logic

the truth of tMEL formulas is defined as:

- $(M, w) \Vdash P \quad P$ is a propositional letter.
- $(M, w) \Vdash F \vee G$ iff $(M, w) \Vdash F$, or $(M, w) \Vdash G$
- $(M, w) \Vdash F \wedge G$ iff $(M, w) \Vdash F$, and $(M, w) \Vdash G$
- $(M, w) \Vdash F \rightarrow G$ iff $(M, w) \nVdash F$, or $(M, w) \Vdash G$
- $(M, w) \Vdash \neg F$ iff $(M, w) \nVdash F$
- $(M, w) \Vdash \mathbf{K}^i F$ iff
  - for all w', $w\mathcal{R}w'$ $(M, w') \Vdash F$ and
  - $\alpha_w(F) \leq i$

- classical tautology,
- $\mathbf{K}^i(\phi \to \psi) \to \mathbf{K}^j\phi \to \mathbf{K}^k\psi$ for $i, j < k$,
- $\mathbf{K}^i A \to \mathbf{K}^j(\mathbf{K}^i A)$ $i < j$ if $A \in \mathcal{A}$
- $\mathbf{K}^i\phi \to \mathbf{K}^j\phi$ $i < j$,
- $\mathbf{K}^i\phi \to \phi$,
- $\mathbf{K}^i\phi \to \mathbf{K}^j(\mathbf{K}^i\phi)$ $i < j$

Logical Bases

- That a tuple $\mathcal{A} = \langle \mathbf{A}, f \rangle$ is a *logical base* informally means that $\mathbf{A}$ is a collection of logical truths.

tS4 Logical Bases

A base $\mathcal{A} = \langle \mathbf{A}, f \rangle$ is tS4 logical if one of the following is true:

- $\mathcal{A}$ is empty
- $\mathcal{A}$ is over another tS4 logical base $\mathcal{B}$, that is, for every $\phi \in \mathcal{A}$, $\models_{tS4(\mathcal{B})} \phi$
- Given an ascending tS4 logical bases $\{\mathcal{A}_i\}_{i \in \mathbb{N}}$, where $\mathcal{A}_{i+1}$ is over $\mathcal{A}_i$ for each $i \in \mathbb{N}$, $\mathcal{A} = \bigcup \mathcal{A}_i$

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- Several Logical Bases $\mathcal{A} = \langle \mathbf{A}, f \rangle$:
  - $\mathcal{A}$ is empty.
  - $\mathcal{A}$ is the collection of classical tautology.
  - $\mathcal{A}$ is the collection of intuitionistic tautology.
  - $\mathcal{A}$ is comprehensive: if for every tS4($\mathcal{A}$)-valid formula ϕ, $\phi \in \mathbf{A}$.
  - $\mathcal{A}$ is principal: if $\mathcal{A}$ is comprehensive and f is the constant function 0.
  - $\mathcal{A}$ is full: if $\vDash_{tS4(\mathcal{A})} \phi$, then $\vDash_{tS4(\mathcal{A})} \mathbf{K}^i\phi$, for some $i \in \mathbb{N}$.

tS4($\mathcal{A}$) Axiom Systems

classical propositional axiom schemes	
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$\mathbf{K}^i A \to \mathbf{K}^j(\mathbf{K}^i A) \quad i < j$ if $A \in \mathcal{A}$	(Deduction by $\mathcal{A}$-Epistemization)
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$\vdash \mathbf{K}^i A$, if $A \in \mathcal{A}$ and $f(A) \leq i$	($\mathcal{A}$-Epistemization)

Theorem (Completeness Theorem)

Given a tS4 logical base $\mathcal{A}$, $\vdash_{tS4(\mathcal{A})} \phi$ if and only if $\models_{tS4(\mathcal{A})} \phi$.

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$\vdash \psi$, if $\vdash \phi \to \psi$ and $\vdash \phi$
$\vdash \mathbf{K}^i A$, if $A \in \mathcal{A}$ and $f(A) \le i$

- When $\mathcal{A}$ is empty, the A2 axiom and the R2 rule are void.

classical propositional axiom schemes

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- When $\mathcal{A}$ is empty, the A2 axiom and the R2 rule are void.
- When $\mathcal{A}$ is comprehensive, the clause "$A \in \mathcal{A}$" can be replaced by "$\vdash A$",
- When the base is principal, "and $f(A) \leq i$" can be removed.

classical propositional axiom schemes

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- When $\mathcal{A}$ is empty, the A2 axiom and the R2 rule are void.
- When $\mathcal{A}$ is comprehensive, the clause "$A \in \mathcal{A}$" can be replaced by "$\vdash A$",
- When the base is principal, "and $f(A) \leq i$" can be removed.
- When $\mathcal{A}$ is axiomatically appropriate,
  the clause "$A \in \mathcal{A}$" can be replaced by "A is an axiom"

Definition

A tS4 logical base $\mathcal{A}$ is *axiomatically appropriate* if it contains all axiom instances of the schemes listed in the above system.

Lemma (similar to the internalization theorem for LP)

If $\mathcal{A}$ is axiomatically appropriate, then $\mathcal{A}$ is full.

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Lemma (similar to the internalization theorem for LP)

If $\mathcal{A}$ is axiomatically appropriate, then $\mathcal{A}$ is full.

That is, if $\mathcal{A}$ is axiomatically appropriate and $\vdash_{tS4(\mathcal{A})} \phi$, then $\vdash_{tS4(\mathcal{A})} \mathbf{K}^i\phi$ for some i.

- If $\mathcal{A}$ is full, $$\cfrac{\cfrac{\vdash_{tS4(\mathcal{A})} \phi \rightarrow \psi}{\vdash_{tS4(\mathcal{A})} \mathbf{K}^i(\phi \rightarrow \psi)}}{\vdash_{tS4(\mathcal{A})} \mathbf{K}^0\phi \rightarrow \mathbf{K}^j\psi}$$

- If $\mathcal{A}$ is full and $\phi_0, \ldots, \phi_n \vdash \psi$, we can derive $\mathbf{K}^0\phi_0, \ldots, \mathbf{K}^0\phi_n \vdash_{tS4(\mathcal{A})} \mathbf{K}^i\psi$ for some i.

- Temporalization Theorem
- Given a *full* tS4-logical base $\mathcal{A}$, ϕ is S4 valid (a S4 theorem) if and only if we can find suitable numerical labels *i* for each modal subformulas of the form $\mathbf{K}\psi$ such that when we substitute $\mathbf{K}^i\psi$ for $\mathbf{K}\psi$, the result formula ϕ^τ is S4($\mathcal{A}$) valid (a S4($\mathcal{A}$) theorem)

tK

classical propositional axiom schemes

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$\mathbf{K}^i F \to \mathbf{K}^j(\mathbf{K}^i F) \quad i < j$

$\mathbf{K}^i F \to F$

$\vdash \psi$, if $\vdash \phi \to \psi$ and $\vdash \phi$

$\vdash \mathbf{K}^i A$, if A is an axiom and $f(A) \leq i$

More Logics

	R	$\mathfrak{A}$
tK	no condition	normal
tKT	reflexive	normal
$tK4$	transitive	positive regular
$tK5$	euclidean	negative regular
$tKT5$	reflexive, and euclidean	negative regular
$tK45$	transitive and euclidean	positive and negative regular
$tKT45$	transitive, reflexive, euclidean	positive and negative regular

Thanks!